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# Dynamical noncommutativity

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Abstract: We present a model of Moyal-type noncommutative algebra with time-depending noncommutativity parameter and the exact gauge invariant action for the U(1) noncommutative gauge theory. We briefly discuss the results of the analysis of plane-wave propagation in a regime of a small but rapidly changing noncommutativity.

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## 1. Introduction

There are many arguments that the picture of smooth space-time should change when approaching the Planck scale. Similarly as in the case of classical and quantum physics, when classical limit is recovered from the quantum description for macroscopic objects, we are probably in a need of a similar approach to describe in a consistent way the quantum theory of gravity. Apart from the model-independent considerations, such as black hole formation limits in quantum mechanics [1] leading to uncertainty relations for spacetime coordinates, there are interesting models in which noncommutativity appears explicitly (see [2] and reviews [4, 5] for an exhaustive list of string-related references).

It should be stressed that there is more to noncommutative geometry than the particular type of the Moyal deformation, which appears in the above mentioned string-motivated theories. In particular, quite appealing are the appearances of finite noncommutative geometry in the Standard Model, where the Higgs field is recovered as the connection field in the finite-geometry [6]. Matrix geometries like fuzzy manifolds are another possibility both as approximations or effective geometries at some energy scales [7]. On the other hand, quantum deformations (see, for instance, [8] for a review of links with physics) offer a vast realm of models with well-defined deformations of symmetries.

In this paper we shall suggest a simple model, which extends the idea of a static or a global noncommutativity towards the time-dependent or a local one. Clearly, in general, such models will not provide an effective description of a flat space time. However, we believe that while considering real physical models, which are not static, like, for example, in cosmology, in black hole formation or even particle interactions, we need to consider a possibility that the effective noncommutativity, which might arise in the picture could be as well space- and time-dependent.



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## 2. Moyal deformation and its generalizations

We shall begin by recalling different formulations of the Moyal deformations of $\mathbb{R}^4$ (though clearly it could be realised in any dimension).

On the level of generators the algebra could be described as defined by (selfadjoint, at least formally) generators $x^\mu$ and relations:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} , \tag{2.1}$$

where $\theta^{\mu\nu}$ is a real, constant, antisymmetric matrix. This provides an algebraic description, though restricts it, basically, to the polynomial algebra. Therefore, it is more convenient to consider the deformation on the level of smooth ($C^\infty$) functions, where it becomes a nonlocal deformation of the usual product on the vector space of functions:

$$f(x) * g(x) = e^{\frac{1}{2} i \partial^x_\mu \theta^{\mu\nu} \partial^y_\nu} f(x) g(y)_{|x=y} . \tag{2.2}$$

Although for practical reasons one works mostly with smooth functions we may as well find out [11] that the deformation is defined as well for the continuous (and bounded) functions, using oscillatory integrals:

$$f(x) * g(x) = \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} d^4z \, d^4y \, f(x + \theta(z)) g(x + y) e^{2\pi i (y \cdot x)} . \tag{2.3}$$

where $\theta(z) = \theta^{ij} z_j$ and $x \cdot y$ is the standard scalar product of two vectors.

It is a nice exercise that both (2.2) and (2.3) give (2.1) when applied to monomials.

The differential structures remain almost the same as in the undeformed case, in fact one can extend the linear isomorphism between the deformed and undeformed functions onto the entire differential complex.

### 2.1 The time-dependent noncommutativity

It was observed [2, 3] that in the string theory the effective geometry of space time generated by strings in the background of a constant $B$-field yields the Moyal deformation. As this corresponds to a flat brane embedded in a flat background space one may ask what picture might arise from considering curved branes or curved backgrounds. The modifications of the Moyal star product in this case were studied in details in [9], where the it was shown that in the topological limit the deformation is given by the Kontsevich star product in the symplectic case or by the nonassociative version of the product in the most general situation:

$$\begin{aligned} f * g = {} & f\, g + \frac{1}{2}\alpha^{ab}\partial_a f\, \partial_b g - \frac{1}{8}\alpha^{ac}\alpha^{bd}(\partial_a\partial_b f)(\partial_c\partial_d g) - \\ & - \frac{1}{12}\alpha^{ad}\partial_d\alpha^{bc}(\partial_a\partial_b f\, \partial_c g - \partial_b f\, \partial_a\partial_c g) + \mathcal{O}(\alpha^3) , \end{aligned} \tag{2.4}$$

where $\alpha^{ab}$ depends on the combination of $B$ and $F$ fields on the brane [9]. Generally the deformation (2.4) is quite complicated, however, there exist some special cases in which the modification of the commutation relations is minor.



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One of the simplest possible models arises when we assume that $\alpha^{ab}$ depends only on one variable and, in addition, its component in this direction vanishes:

$$\alpha^{ab}\partial_b\alpha^{cd} = 0\,.$$

A particular solution of this condition is:

$$\begin{aligned}\alpha^{\mu 0} = \alpha^{0\mu} &= 0\,,\\ \partial_i\alpha^{jk} &= 0\,,\end{aligned}$$

which leads to the construction of a time-dependent noncommutativity of space coordinates:

$$\begin{aligned}[][x^i, x^j] &= i\theta^{ij}(t)\,,\\ [x^i, t] &= 0\,,\end{aligned} \tag{2.5}$$

where $\theta^{ij}(t)$ is a smooth funcion of $t$ valued in antisymmetric real matrices. For arbitrary smooth functions (identified as elements of the vector space of the deformed algebra) the product could be written as:

$$f(x) * g(x) = e^{\frac{1}{2}i\partial_i^x\theta^{ij}(t)\partial_j^y} f(x)g(y)_{|x=y}\,. \tag{2.6}$$

It could be argued that such "dynamical noncommutative manifolds" are as good object as the normal manifolds or deformed ones [12]. They might give an insight as to whether noncommutavity could be treated within nontrivial dynamical systems and whether it might have evolved with the cosmological evolution or, for instance, it could accompany the creation of black holes.

Let us consider functions on $\mathbb{R}^4$, which, satisfy (for the polynomials) have the commutation relations (2.5). In our case, when we consider a general construction, the function $\theta$ can *a priori* be arbitrary, in particular, it could interpolate between the highly noncommutative and commutative regime.

What changes in comparison with the $\theta = \text{const}$ case is the differential calculus:

$$\begin{aligned} t\,dx^i &= dx^i\,t\,,\\ x^i\,dt &= dt\,x^i\,,\\ [x^i, dx^j] &= \left(\frac{1}{2}i\dot{\theta}^{ij} + A^{ij}(t)\right)dt\,,\end{aligned} \tag{2.7}$$

where $A^{ij} = A^{ji}$ is any symmetric matrix. The change is only on the level of commutations between the forms and the functions, as the products of the generating forms do not change at all:

$$dx^\mu\,dx^\nu + dx^\nu\,dx^\mu = 0\,. \tag{2.8}$$

The commutation rules between differentials and the arbitrary functions are:

$$f(x)\,dx^j = dx^j\,f(x) + \left(\frac{1}{2}i\dot{\theta}^{ij} + A^{ij}(t)\right)(\partial_i f(x))\,dt\,. \tag{2.9}$$



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We use the usual $\mathbb{R}^4$ partial derivatives $\partial_\mu$, however, note that due to noncomutativity (2.6) they no longer must obey the Leibniz rule, in particular we observe that:

$$\partial_t(f*g) = (\partial_t f)*g + f*(\partial_t g) + \left(\frac{1}{2}i\dot{\theta}^{ij} + A^{ij}(t)\right)(\partial_i f)*(\partial_j g)\,. \tag{2.10}$$

As a final remark of this section let us observe that the standard conjugation on the vector space of complex-valued smooth functions on $\mathbb{R}^4$ is still a well defined conjugation of the deformed algebra, i.e.:

$$\overline{(f*g)} = \bar{g}*\bar{f}\,, \tag{2.11}$$

and that it extends on the differential algebra so that all $dx^\mu$ are selfadjoint.

It is easy to see that the considered time-dependent deformation have the same trace (integral) as the usual one, therefore we shall have:

$$\int f*g = \int d^4x\, f(x)g(x)\,. \tag{2.12}$$

and

$$\int (\partial_\mu f) = 0\,. \tag{2.13}$$

Note that in the commutation relations for differentials there are two terms of different origin: the derivative of $\theta^{ij}$, which is connected to the dynamical deformation of space-time and the term proportional to an arbitrary symmetric matrix $A^{ij}(t)$. The latter is in no way related to our deformation, hence we shall put $A^{ij}(t) = 0$ throughout the rest of this paper.

## 3. Klein-Gordon equation

Although there is no natural Hodge star on the differential algebra, which we have constructed in the previous section (at least not as a bimodule map, still a left-module Hodge map exists and is a straightforward generalisation from the undeformed case) we have a natural metric understood as a bimodule map

$$g : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \mathcal{A}\,,$$

given by (for instance):

$$g(dx^i, dx^j) = -\delta^{ij}\,, \qquad g(dx^i, dt) = 0 = g(dt, dx^i)\,, \qquad g(dt, dt) = 1\,.$$

Then we might consider the dynamical scalar field action of the form

$$S_\Phi = \int g(d\Phi, d\Phi^*) = \int \eta^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi)^*\,, \tag{3.1}$$

where $\eta$ denotes the tensor components of the above defined flat Minkowski metric $(+,-,-,-)$. The resulting wave equation is formally the same:

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\Phi = 0\,. \tag{3.2}$$

though we must take into account that $\partial_t$ is not a derivation.



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It appears, however that on functions, which depend only on linear functions of spatial coordinates, that is: $f = f(t, k_i x_i)$ all partial derivatives are derivations, we verify it explicitly on $(k_i x_i)^2$:

$$\partial_0 (k_i x_i)^2 = \frac{1}{2}\dot{\theta}^{ij} k_i k_j = 0\,,$$

because $\theta$ is antisymmetric. Extending the results on all polynomials (by induction) we might conclude that the canonical solution of the Klein-Gordon equation on the dynamical noncommututative deformation, which we present is the same as in the undeformed case $\Phi = \Phi(k_\mu x^\mu)$ for any null vector $k_\mu$, $k^2 = 0$.

## 4. Gauge theory

The principal change in noncommutative gauge theory is the appearance of gauge-field self-interaction terms (nonlocal from the commutative point of view) for the U(1) gauge field theory. A noncommutative gauge connection is an antiselfadjoint one-form $A_\mu dx^\mu$, for which the gauge strength field is $F = dA + A\,A$.

First of all, unlike in the commutative case the components of the gauge potential shall not be imaginary functions: this shall remain true only for the spatial components, whereas the time component shall be composed of an arbitrary imaginary field and have a part depending of the other fields:

$$A_0 + (A_0)^* = \frac{1}{2} i\dot{\theta}^{ij}(\partial_i A_j)\,. \tag{4.1}$$

Moreover, in the case of time-dependent noncommutativity we have, in addition to the appearance of the usual nonlinear terms, the problem of the proper definition of the gauge invariant Yang-Mills action.

Since, due to (2.9) functions do not commute with differentials, the rules of gauge transformations for the components of the field strength are:

$$F'_{ij} = U^\dagger * F_{ij} * U\,, \tag{4.2}$$

$$F'_{0i} = U^\dagger * F_{0i} * U + \frac{1}{2} i\dot{\theta}^{kj} U^\dagger * F_{ij} * (\partial_k U)\,, \tag{4.3}$$

where

$$\begin{aligned} F_{ij} &= \partial_i A_j - \partial_j A_i + [A_i, A_j]\,, \qquad i < j\,, \\ F_{0i} &= \partial_0 A_i - \partial_i A_0 + [A_0, A_i] - \frac{1}{2} i\left(\dot{\theta}^{mn} A_n * (\partial_m A_i)\right). \end{aligned} \tag{4.4}$$

Clearly, $F_{ij}F^{ij}$ remains gauge covariant but $F_{0i}F^{0i}$ is not. However, let us observe that we can easily find a gauge covariant expression $\widetilde{F}_{0i}$:

$$F_{0i} - \frac{1}{2} i\dot{\theta}^{kj} F_{ij} * A_k\,. \tag{4.5}$$

The additional term transforms like:

$$\frac{1}{2} i\dot{\theta}^{kj} * U^\dagger * F_{ij} * A_k * U + \frac{1}{2} i\dot{\theta}^{kj} * U^\dagger * F_{ij} * (\partial_k U)\,,$$

and exactly cancels the corresponding additional term in the transformation rule of $F_{i0}$. Then the term, which would contribute to the gauge-invariant action, $\widetilde{F}_{0i}$, reads:

$$\widetilde{F}_{0i} = \partial_0 A_i - \partial_i A_0 + [A_0, A_i] - \frac{1}{2} i\dot{\theta}^{kj} \left( F_{ij} * A_k + A_j * (\partial_k A_i) \right).$$

Finally, we can propose the gauge invariant action:

$$S = \int \sum_{i<j} (F_{ij} * F^*_{ij}) - \sum_i (\widetilde{F}_{0i} * \widetilde{F}^*_{0i}), \tag{4.6}$$

which, in addition to the standard terms of classical electrodynamics contains the corrections resulting from the dynamical character of noncommutativity of space. We shall analyse these corrections in more detail for the example of plane-waves.

## 5. Physical effects in dynamical noncommutativity

In this section we shall briefly discuss the potential observable physical consequences of the dynamical noncommutativity. In what follows we shall always assume that the parameter $\theta$, which determines the strength of noncommutativity is negligible, however, its time variation is not. Therefore we shall skip all the terms proportional to $\theta$ and keep only the lowest order corrections in $\dot{\theta}$. This corresponds to the intuitive picture of short rapid "bursts" of noncommutativity in the past history of the universe. We shall concentrate on small perturbations to the propagation of electromagnetic waves caused by such events, using the earlier derived action (4.6) and assuming that the gauge potential is of the plane-wave form.

First, note that we have to take into account the nonlinear form of the anti-selfadjoint condition (4.1), thus leading to:

$$\begin{aligned} A_i &= i p_i f(\omega t + k_i x^i), \\ A_0 &= i p_0 f(\omega t + k_i x^i) - \frac{1}{4} \dot{\theta}^{ij} (k_i p_j) f'(\omega t + k_i x^i). \end{aligned} \tag{5.1}$$

where $p_\mu, k_i$ are fixed (real) vectors and $\omega$ is a (real) constant.

For the components of the field strength we have:

$$\begin{aligned} F_{0i} &= i(\omega p_i - k_i p_0) f' + \frac{1}{2} i f * f' \dot{\theta}^{jk} (k_j p_k p_i) + \frac{1}{4} f'' \dot{\theta}^{jk} (k_j p_k k_i), \\ F_{ij} &= i(k_i p_j - p_i k_j) f'. \end{aligned} \tag{5.2}$$

The gauge-covariant term obtained as a correction of $F_{i0}$ will be:

$$\begin{aligned} \tilde{F}_{0i} &= F_{0i} - \frac{1}{2} i\dot{\theta}^{jk} k_j p_i p_k f' * f \\ &= i(\omega p_i - k_i p_0) f' - \frac{1}{2} i\dot{\theta}^{jk} k_j p_i p_k (f' * f + f * f') + \frac{1}{4} f'' \dot{\theta}^{jk} (k_j p_k k_i). \end{aligned} \tag{5.3}$$



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The action for the plane-wave Ansatz reads:

$$-\left(\vec{k}^2\vec{p}^2 + 2(\omega p_0)(\vec{k}\vec{p}) - (\vec{k}\vec{p})^2 - \omega^2\vec{p}^2 - p_0^2\vec{k}^2\right) - \int (f')^2 -$$
$$-2k_jp_k\left(\omega\vec{p}^2 - p_0(\vec{k}\vec{p})\right)\int \dot{\theta}^{jk} f' * f' * f +$$
$$+o(\dot{\theta})\,, \tag{5.4}$$

where we have omitted terms of higher order in $\dot{\theta}$, and we have used the trace property of the integral (2.12).

Let us concentrate for a while on the consequences of the additional term, which appears in the effective action for the electromagnetic plane wave. First of all, the lowest order local term for the self-interaction is of third order in the field $A$. It vanishes only if the wave is $\dot{\theta}$-polarised, that is if $\dot{\theta}^{ij}k_ip_j = 0$. Since the correction parameter is itself a function of time, of which we have assumed that its derivative is (relatively) significant we cannot treat the additional term as a constant modification, which might have modified the dispersion relation. Instead the highly nonlinear character of the term suggests rather that one would expect the dynamical noncommutativity to induce effects similar to these of nonlinear optics, and, in particular, the frequency doubling of electromagnetic radiation, which is, in principle, measurable.

Above, we have taken only the simple example based on the pure U(1) gauge theory in our model of dynamical noncommutativity, which we interpret as electrodynamic — to see whether there could be some observable consequences. As we can see, even on the pure classical level the answer is positive: dynamical noncommutativity adds some nontrivial corrections leading to nonlinear selfinteractions of electromagnetic potential, while still keeping the gauge invariance of the theory intact. Still, potentially even more interesting effects might appear on the level of interactions between light and matter, in particular, in the possible corrections to the atomic spectra coming from dynamical noncommutativity.

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